

On black strings & branes in Lovelock gravity

David Kastor^a and Robert Mann^{bc}

^a*Department of Physics, University of Massachusetts
Amherst, MA 01003, U.S.A.*

^b*Department of Physics, University of Waterloo
200 University Avenue West, Waterloo, Ontario, Canada, N2L 3G1*

^c*Perimeter Institute for Theoretical Physics
35 Caroline St. N., Waterloo, Ont. Canada*

E-mail: kastor@physics.umass.edu, rbmann@sciborg.uwaterloo.ca

ABSTRACT: It is well known that black strings and branes may be constructed in pure Einstein gravity simply by adding flat directions to a vacuum black hole solution. A similar construction holds in the presence of a cosmological constant. While these constructions fail in general Lovelock theories, we show that they carry over straightforwardly within a class of Lovelock gravity theories that have (locally) unique constant curvature vacua.

KEYWORDS: Classical Theories of Gravity, Black Holes.

Contents

1.	Introduction	1
2.	LUV theories	3
3.	Black strings and branes in LUV theories	5
4.	Concluding remarks	9

1. Introduction

It is straightforward to construct uncharged black strings and branes in Einstein gravity. Starting from a Schwarzschild black hole in D -dimensions, for example, one obtains a black string in $(D + 1)$ -dimensions simply by adding a flat direction to the metric. Despite the seemingly trivial nature of this construction, the phenomenology associated with black strings, beginning with the Gregory-Laflamme instability [1] , has turned out to be surprisingly rich (see reference [2] for a review). One can similarly construct black p -branes by adding additional flat directions, while rotating black branes can be obtained by adding flat directions to the D -dimensional Myers-Perry metrics [3].

Lovelock gravity theories [4] are fascinating extensions of general relativity that include higher curvature interactions. The Lagrangian density for Lovelock gravity in D spacetime dimensions can be written $\mathcal{L} = \sum_{k=0}^{\lfloor d/2 \rfloor} c_k \mathcal{L}_k$, where

$$\mathcal{L}_k = \frac{1}{2^k} \sqrt{-g} \delta_{c_1 \dots c_k d_1 \dots d_k}^{a_1 \dots a_k b_1 \dots b_k} R_{a_1 b_1}{}^{c_1 d_1} \dots R_{a_k b_k}{}^{c_k d_k}, \tag{1.1}$$

and $\mathcal{L}_0 = \sqrt{-g}$. The δ symbol above denotes the totally antisymmetric product of $2k$ Kronecker deltas, normalized to take values 0 and ± 1 . The term \mathcal{L}_0 gives the cosmological term in the action, while $\mathcal{L}_1 = \sqrt{-g}R$ gives the Einstein term. The curvature squared term, known as the Gauss-Bonnet term, may be expanded to give $\mathcal{L}_2 = \sqrt{-g}(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2)$. In $D = 4$ this term can be written as a divergence and does not contribute to the equations of motion. Similarly the term \mathcal{L}_k is the Euler density in $D = 2k$ and hence contributes to the equations of motion only for dimensions $D > 2k$. Lovelock theories are distinguished, among the much larger class of general higher curvature theories, by having field equations involving not more than second derivatives of the metric. Consequently, Lovelock gravity theories are free from many of the pathologies that plague general higher derivative gravity theories.

The black hole solutions of Lovelock gravity have been studied beginning with the work of [5–7] some 20 years ago. One would also like to know the black string and black brane solutions, *e.g.* in order to study the effects of higher curvature interactions on black string phenomenology. A number of workers have tried to find such solutions. However, one immediately encounters the fact that adding flat directions does not work in the general Lovelock theory. This was noted for Gauss-Bonnet gravity in reference [8]. Thus far, analytic solutions for black branes in general Lovelock theories have not been found, and workers have turned to other methods. In particular, black strings in 5-dimensional Gauss-Bonnet gravity were studied numerically in reference [9] and more general black branes in this theory were studied via near horizon and far field expansions in reference [10].

In this paper, we show that black brane solutions to Lovelock gravity theories including higher curvature terms may, in fact, be simply constructed, but only within a certain class of Lovelock theories. This class of theories has the following property. Assume that \mathcal{L}_p is the highest order term in the Lagrangian, *i.e.* that the coefficients c_k vanish for $k > p$. Depending on the values of the nonzero coefficients in the Lagrangian, it then turns out that the theory may have up to p distinct constant curvature vacuum solutions [5][6][7]. The different values that the constant curvature may take are the roots of a p th order polynomial. There will, of course, generally be p roots, but only real values of the curvature are considered to be physical. The coefficients in the Lovelock Lagrangian may be tuned such that there are p real roots and that all these roots coincide. The theory then has a (locally) unique constant curvature vacuum solution. We will refer to these as LUV theories - standing for Lovelock-Unique-Vacuum. We show that LUV theories have simple black brane solutions.

The simplest LUV theory with non-trivial dynamics is Einstein gravity with a cosmological constant Λ , and our results on brane solutions will be, roughly, that what works in this theory works in all LUV theories. We can first consider Einstein gravity with $\Lambda = 0$. In this case, as described above, brane solutions are simply obtained by adding flat directions. Our first result will be that adding flat directions also works in LUV theories, in the limit that the curvature of the vacuum has been tuned to zero. Taking this limit of a LUV theory with highest order interaction \mathcal{L}_p sends the coefficients of all the lower order terms to zero and we have simply $\mathcal{L} = \mathcal{L}_p$. We will call these pure Lovelock gravity theories, with pure Einstein gravity as the first non-trivial example.

If we now consider Einstein gravity with $\Lambda \neq 0$, additional directions may still be added, but they are no longer flat, *i.e.* the new metric must have nontrivial coordinate dependence on the new direction. A good example of this is the AdS black string of reference [11], which is given by

$$ds^2 = \frac{l^2}{z^2} \left(dz^2 - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\Omega^2 \right) \tag{1.2}$$

where l is the radius of curvature of 5-dimensional AdS. This differs from the black string of pure Einstein gravity in the overall conformal rescaling by the function l^2/z^2 of the new coordinate z . Note also that the 4-dimensional ‘seed’ metric for the construction,

the Schwarzschild metric in this case, is a solution to Einstein gravity with vanishing cosmological constant. More generally the 4-dimensional Schwarzschild metric in (1.2) may be replaced by any Ricci flat metric. We will show below that in LUV theories with nonzero vacuum curvature, new directions may be added in a similar way. The metric will generally be conformally scaled by a function of the new coordinate. The lower dimensional seed metric can be a solution to a LUV theory with a different value of the vacuum curvature, possibly zero as in the metric (1.2) above.

LUV theories and their black hole solutions have been extensively discussed in reference [12]. It is argued there that LUV theories have a number of nice properties that distinguish them as possibly the most physically relevant Lovelock theories. We note [12] that in odd dimensions $D = 2p + 1$ the LUV theory with highest term \mathcal{L}_p , which is the highest non-trivial term, corresponds to Chern-Simons gravity¹, while in even dimensions the LUV theory that includes the highest non-trivial interaction follows from a Born-Infeld type action.

2. LUV theories

In this section we recount some useful results in Lovelock gravity, focusing in particular on the LUV theories. The equations of motion following from the Lovelock Lagrangian $\mathcal{L} = \sum_{k=0}^r c_k \mathcal{L}_k$, with $r \leq [d/2]$ have the form $\mathcal{G}_{ab} = 0$ where

$$\mathcal{G}^a_b = \sum_{k=0}^r c_k \delta_{be_1 \dots e_k f_1 \dots f_k}^{\alpha c_1 \dots c_k d_1 \dots d_k} R_{c_1 d_1}^{e_1 f_1} \dots R_{c_k d_k}^{e_k f_k}. \tag{2.1}$$

For the purposes of studying LUV theories, it is also useful to write \mathcal{G}_{ab} in an alternative form which we will make use of below,

$$\mathcal{G}^a_b = \alpha_0 \delta_{be_1 \dots e_r f_1 \dots f_r}^{\alpha c_1 \dots c_r d_1 \dots d_r} \left(R_{c_1 d_1}^{e_1 f_1} + \alpha_1 \delta_{c_1 d_1}^{e_1 f_1} \right) \dots \left(R_{c_r d_r}^{e_r f_r} + \alpha_r \delta_{c_r d_r}^{e_r f_r} \right). \tag{2.2}$$

The original form of the equations of motion can then be recovered through repeated applications of the identity

$$\delta_{b_1 \dots b_p}^{\alpha_1 \dots \alpha_p} \delta_{a_{p-1} a_p}^{b_{p-1} b_p} = 2(D - (p - 1))(D - (p - 2)) \delta_{b_1 \dots b_{p-2}}^{\alpha_1 \dots \alpha_{p-2}} \tag{2.3}$$

The coefficients c_k are given by sums of products of the parameters α_k in (2.2). The precise relation is given in reference [12]. Inverting this relation to get the α_k 's in terms of the c_k 's requires solving a polynomial equation of order r . Hence the α_k 's are generally complex parameters.

For our purposes, it is convenient to take the parameters α_k with $k = 0, 1, \dots, r$ to be real valued and regard the coefficients c_k in the Lagrangian to be determined by them. It is then clear from the expression (2.2) for \mathcal{G}_{ab} that if the parameters α_k with $k = 1, \dots, r$ are all distinct, then the theory will have r distinct constant curvature solutions, with Riemann tensors given respectively by

$$R_{ab}{}^{cd} = -\alpha_k \delta_{ab}^{cd}, \quad k = 1, \dots, r \tag{2.4}$$

¹Further properties of these theories are discussed in references [13][14].

The LUV theories discussed above result from setting all the parameters α_k with $k = 1, \dots, r$ equal to a common value α . There is then, at least locally, a unique constant curvature vacuum. If we further set $\alpha = 0$, we get a pure Lovelock theory with Lagrangian $\mathcal{L} = \alpha_0 \mathcal{L}_r$, which has flat spacetime as its unique constant curvature vacuum.

The static, spherically symmetric solutions to Lovelock gravity in D -dimensions can be written in the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_n^2 \tag{2.5}$$

where $n = D - 2$. The equations of motion in the form (2.3) resemble an r th order polynomial equation of order. It is then not surprising that the metric function $f(r)$ involves solving an r th order algebraic equation [5–7] (see also [15] for a detailed analysis of the general case). We will examine the solutions in Gauss-Bonnet gravity, in which $r = 2$, in some detail. In this case the solutions for the metric function $f(r)$ has two branches, given by

$$f_{\pm}(r) = 1 - \frac{r^2}{2} \left(-(\alpha_1 + \alpha_2) \pm \sqrt{(\alpha_1 - \alpha_2)^2 + \frac{4}{r_0^4} \left(\frac{r_0}{r}\right)^{D-1}} \right). \tag{2.6}$$

As r tends to infinity the branches f_+ and f_- approach constant curvature metrics with curvatures $-\alpha_2$ and $-\alpha_1$ respectively. If we take $\alpha_2 = 0$, then the solutions in the f_+ brance are asymptotically flat. Keeping the leading order correction to the flat metric in this case, we have as r tends to infinity

$$f_+ \simeq 1 - \frac{1}{\alpha_1 r_0^2} \left(\frac{r_0}{r}\right)^{D-3}. \tag{2.7}$$

This is the correct fall-off for a black hole of finite ADM mass in D -dimensions. However, the metric differs from the Schwarzschild metric at smaller scales.

We can take the LUV limit of the Gauss-Bonnet theory by setting $\alpha_1 = \alpha_2 = \epsilon/l^2$, with $\epsilon = \pm 1$. The expressions for the metric functions simplify considerably in this limit [12] to be

$$f_{\pm}(r) = 1 + \epsilon \frac{r^2}{l^2} \mp \left(\frac{r_0}{r}\right)^{\frac{(D-5)}{2}} \tag{2.8}$$

For $D > 5$, the branch f_+ then gives black holes in (anti-)deSitter, while the branch f_- gives naked singularities. In each case, the falloff to the vacuum is slower than in Schwarzschild-(A)dS, and the ADM mass would therefore be infinite. However, it was shown in [16][12] that an appropriate definition of the mass in LUV theories gives a finite result.

One can further take the limit $l \rightarrow \infty$ to get pure Gauss-Bonnet theory. We then have simply

$$f_{\pm}(r) = 1 \mp \frac{|C|}{r^{\frac{(D-5)}{2}}}. \tag{2.9}$$

For $D > 5$ the two branches again yield either black holes, or naked curvature singularities. The case $D = 5$, however, is particularly interesting. In this case the metric function is simply a constant. Rescaling the time and radial coordinates, the metric can be written as

$$ds^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \alpha^2 \tilde{r}^2 d\Omega_3^2. \tag{2.10}$$

where $\alpha < 1$ for the f_+ corresponding to 3-spheres with missing solid angle, while $\alpha > 1$ for f_- corresponding to an excess of solid angle. In both cases there is a conical singularity at the origin. Other solutions to Lovelock theory with conical singularities at the origin (but with a different geometry in the constant (r, t) section) were obtained recently [17].

Note that $D = 5$ is the lowest dimension for which the Gauss-Bonnet term is dynamically relevant. The situation is similar [12] for all pure Lovelock theories, $\mathcal{L} = \mathcal{L}_p$, in dimension $D = 2p + 1$. The static, spherically symmetric solutions have spheres S^{2p-1} with missing (or excess) solid angle. The first example with $p = 1$, *i.e.* pure Einstein gravity, in $D = 3$ is a familiar one. Mass in this theory corresponds to point conical defects in an otherwise flat spacetime. In the general case spacetime is not flat, but satisfies the pure Lovelock field equations.

3. Black strings and branes in LUV theories

In pure Einstein gravity, we know that we can construct black brane solutions by adding flat directions to any black hole solution. More precisely, if we have a spacetime metric $\hat{g}_{\mu\nu}$ that solves the vacuum Einstein equations in D -dimensions, then the metric

$$ds^2 = dz^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu \tag{3.1}$$

solves the vacuum field equations in $D + 1$ dimensions. If $\hat{g}_{\mu\nu}$ is the metric for a black hole with horizon topology Σ , then the new metric is a black string with horizon topology $\Sigma \times R^1$. The procedure can be iterated to generate black branes with any number of flat spatial dimensions tangent to the horizon.

It is straightforward to check that this simple construction does not work in general Lovelock theories. In fact, it fails already in Einstein gravity with a nonzero cosmological constant. Consider Einstein's equations $R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 0$ for the $(D + 1)$ -dimensional metric (3.1). Assume that $\hat{g}_{\mu\nu}$ satisfies these same equations. Note that all components of the Riemann tensor of the metric (3.1) with one, or more, z indices necessarily vanish. In particular, this implies that $R_{\mu\nu} = \hat{R}_{\mu\nu}$ and $R = \hat{R}$ where the hatted quantities are those computed from $\hat{g}_{\mu\nu}$. The $\mu\nu$ components of the field equations are then identical to those for $\hat{g}_{\mu\nu}$ which are satisfied by assumption. The zz component of the field equations, however, requires $\frac{1}{2}\hat{R} - \Lambda = 0$, which conflicts with the trace of the lower dimensional field equations which implies $\frac{2-D}{2}\hat{R} + D\Lambda = 0$. We then conclude that the $(D + 1)$ -dimensional metric (3.1) satisfies the field equations only for $\Lambda = 0$, pure Einstein gravity.

Our first new result is to observe that, while adding flat directions does not generally work in Lovelock gravity, it does work in all pure Lovelock theories, with Lagrangians $\mathcal{L} = \mathcal{L}_p$. The result above for pure Einstein gravity can then be seen as the particular case $p = 1$ of this more general statement. If $\hat{g}_{\mu\nu}$ solves the field equations of the pure Lovelock theory in D -dimensions, then the metric (3.1) solves the pure Lovelock field equations in $D + 1$ dimensions. This is easily seen from the field equations (2.1), with only c_p nonzero, in

the following way. Note that as before all the components of the Riemann curvature tensor with one, or more, z indices necessarily vanish. The field equations $\mathcal{G}^\mu{}_\nu = 0$ are then again identical to the D -dimensional field equation $\hat{\mathcal{G}}^\mu{}_\nu = 0$ for $\hat{g}_{\mu\nu}$ and hence are satisfied by assumption. The equation $\mathcal{G}^z{}_z = 0$ is given by the trace $\hat{\mathcal{G}}^\mu{}_\mu$ and hence also vanishes. Note that, as in the pure Einstein case, the trace of the pure Lovelock field equations for $\mathcal{L} = \mathcal{L}_p$ is equal to $\mathcal{L}_p/\sqrt{-\hat{g}}$ times a constant.

This result leads to some intriguing possibilities. In pure Einstein theory, flat directions can be added to the point-like conical defects of $D = 3$ gravity to get cosmic strings in $D = 4$ and branes in higher dimensions, which are surrounded by circles with deficit (surfeit) angle. These defects play important roles in, for example, the Weyl solutions and, in particular, in the C-metric. There is great interest in higher dimensional generalizations of these spacetimes (see e.g. [18, 19]). If we take the case of the pure Gauss-Bonnet gravity, \mathcal{L}_2 , then in $D = 5$ we have a static, spherically symmetric solution with missing (excess) angle on an S^3 . If we add one flat direction, we get a string in $D = 6$ surrounded by an S^3 with missing (excess) solid angle. It is tempting to think that this string might serve as a starting point for finding analogues of the Weyl solutions or the C-metric in $D = 6$.

The next step is to attempt to find a construction of black strings and branes that allows us to add back in the lower order terms in the Lovelock Lagrangian. Here, we are able to make some partial progress by again taking results from Einstein gravity as a guide. If one considers Einstein gravity with a negative cosmological constant $\Lambda = -1/l^2$, then we know that if a metric $\hat{g}_{\mu\nu}$ in D dimensions is Ricci-flat, then the metric

$$ds^2 = \frac{l^2}{z^2}(dz^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu) \tag{3.2}$$

solves the field equations in $D + 1$ dimensions (see e.g. [11]). If we try this in Lovelock theory, we find the following result. If $\hat{g}_{\mu\nu}$ solves the field equations of pure D -dimensional Lovelock gravity with Lagrangian \mathcal{L}_p , then the metric (3.2) solves the field equations of the $(D + 1)$ -dimensional LUV theory, also with the highest interaction term \mathcal{L}_p , but with $\alpha = 1/l^2$.

To keep the formulas manageable, we will focus on the case $p = 2$ of Gauss-Bonnet gravity. We then indicate how the proof straightforwardly generalizes to higher order LUV theories. Our D -dimensional spacetime $\hat{g}_{\mu\nu}$ is then assumed to satisfy the equations of motion of pure Gauss-Bonnet gravity

$$0 = \delta_{\nu\sigma_1\dots\sigma_4}^{\mu\rho_1\dots\rho_4} \hat{R}_{\rho_1\rho_2}{}^{\sigma_1\sigma_2} \hat{R}_{\rho_3\rho_4}{}^{\sigma_3\sigma_4} \tag{3.3}$$

where $\hat{R}_{\mu\nu\rho}{}^\sigma$ is the Riemann tensor of $\hat{g}_{\mu\nu}$. If we let g_{ab} denote the metric in (3.2) then its nonzero curvature components are given by

$$R_{\mu z}{}^{\nu z} = -\frac{1}{l^2}\delta_\mu^\nu \quad R_{\mu\nu}{}^{\rho\sigma} = \frac{z^2}{l^2}\hat{R}_{\mu\nu}{}^{\rho\sigma} - \frac{1}{l^2}\delta_{\mu\nu}^{\rho\sigma} \tag{3.4}$$

The field equations of the LUV theory are given by $\mathcal{G}^a_b = 0$. After plugging in the curvature components (3.4) the \mathcal{G}^z_z component of the field equations becomes

$$0 = \delta_{\sigma_1 \dots \sigma_4}^{\rho_1 \dots \rho_4} \left(\frac{z^2}{l^2} \hat{R}_{\rho_1 \rho_2}{}^{\sigma_1 \sigma_2} + \left(\alpha - \frac{1}{l^2} \right) \delta_{\rho_1 \rho_2}^{\sigma_1 \sigma_2} \right) \left(\frac{z^2}{l^2} \hat{R}_{\rho_3 \rho_4}{}^{\sigma_3 \sigma_4} + \left(\alpha - \frac{1}{l^2} \right) \delta_{\rho_3 \rho_4}^{\sigma_3 \sigma_4} \right). \quad (3.5)$$

Having made the choice $\alpha = 1/l^2$, we see that this is simply proportional to the trace of the equations of motion (3.3) for $\hat{g}_{\mu\nu}$. Plugging into the \mathcal{G}^μ_ν components of the field equations gives

$$0 = \delta_{\nu \sigma_1 \dots \sigma_4}^{\mu \rho_1 \dots \rho_4} \left(\frac{z^2}{l^2} \hat{R}_{\rho_1 \rho_2}{}^{\sigma_1 \sigma_2} + \left(\alpha - \frac{1}{l^2} \right) \delta_{\rho_1 \rho_2}^{\sigma_1 \sigma_2} \right) \left(\frac{z^2}{l^2} \hat{R}_{\rho_3 \rho_4}{}^{\sigma_3 \sigma_4} + \left(\alpha - \frac{1}{l^2} \right) \delta_{\rho_3 \rho_4}^{\sigma_3 \sigma_4} \right) \quad (3.6)$$

$$+ 8 \delta_{\nu \sigma_1 \dots \sigma_3}^{\mu \rho_1 \dots \rho_3} \left(\left(\alpha - \frac{1}{l^2} \right) \delta_{\rho_1}^{\sigma_1} \right) \left(\frac{z^2}{l^2} \hat{R}_{\rho_2 \rho_3}{}^{\sigma_2 \sigma_3} + \left(\alpha - \frac{1}{l^2} \right) \delta_{\rho_2 \rho_3}^{\sigma_2 \sigma_3} \right)$$

For $\alpha = 1/l^2$ the first term again reduces to the D -dimensional equations of motion for $\hat{g}_{\mu\nu}$, while the second term simply vanishes. The equations $\mathcal{G}^z_\mu = 0$ are trivially satisfied, and we have thus shown that adding the construction (3.2) of new solutions holds for LUV theories of maximum order $p = 2$ starting with solutions of pure Gauss-Bonnet theory.

This derivation may be straightforwardly extended to higher order LUV theories as well. Rather than present the more lengthy equations that arise in the general case, we will sketch in words how the equations above for the second order case are generalized in the higher order case. Specifically, take the seed metric $\hat{g}_{\mu\nu}$ in (3.2) to be a solution of the Lovelock theory with $\mathcal{L} = \mathcal{L}_p$ and consider the field equations $\mathcal{G}^a_b = 0$ of the p th order LUV theory with $\alpha = 1/l^2$. The equation $\mathcal{G}^z_z = 0$ will closely resemble equation (3.5), but with p of the factors in parenthesis on the right hand side, rather than 2. Setting $\alpha = 1/l^2$ then gives a multiple $(z^2/l^2)^p$ of the trace of the pure Lovelock field equations, which vanish by assumption. The components \mathcal{G}^μ_ν of the field equations will have a form similar to equation (3.6). In this case, there will be p of the curvature factors in parenthesis on the first line of (3.6), while the term on the second line will have a single factor of the form $(\alpha - \frac{1}{l^2})\delta_\sigma^\rho$ and $p - 1$ factors involving the curvature of the seed metric. The first line will again be proportional to the pure Lovelock field equations for the seed metric $\hat{g}_{\mu\nu}$, while the second line will again vanish upon setting $\alpha = 1/l^2$. The components \mathcal{G}^z_μ will again vanish identically. This then establishes the result for general LUV theories.

We now prove a related, but more general result as well. Assume now that the D -dimensional metric $\hat{g}_{\mu\nu}$ solves the equations of motion of LUV theory with maximum interaction \mathcal{L}_2 and curvature parameter β ,

$$0 = \delta_{\nu \sigma_1 \dots \sigma_4}^{\mu \rho_1 \dots \rho_4} \left(\hat{R}_{\rho_1 \rho_2}{}^{\sigma_1 \sigma_2} + \beta \delta_{\rho_1 \rho_2}^{\sigma_1 \sigma_2} \right) \left(\hat{R}_{\rho_3 \rho_4}{}^{\sigma_3 \sigma_4} + \beta \delta_{\rho_3 \rho_4}^{\sigma_3 \sigma_4} \right). \quad (3.7)$$

Now consider a $(D + 1)$ -dimensional metric of the form

$$ds^2 = \frac{1}{f(z)^2} (dz^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu) \quad (3.8)$$

and ask if it can solve the field equations of the same LUV theory, but with vacuum curvature parameter α instead. Our previous result corresponds to the special case that the curvature parameter of the D -dimensional metric vanishes. The curvature of the $(D + 1)$ -

dimensional metric (3.8) is given by

$$R_{\mu z}{}^{\nu z} = (f\partial_z^2 f - (\partial_z f)^2)\delta_\mu^\nu \quad R_{\mu\nu}{}^{\rho\sigma} = f^2\hat{R}_{\mu\nu}{}^{\rho\sigma} - (\partial_z f)^2\delta_{\mu\nu}^{\rho\sigma} \quad (3.9)$$

The field equations for the $(D + 1)$ -dimensional metric (3.8) are again given by $\mathcal{G}^a{}_b = 0$. Upon plugging in the components of the Riemann tensor, the equation $\mathcal{G}^z{}_z = 0$ becomes

$$0 = \delta_{\sigma_1\dots\sigma_4}^{\rho_1\dots\rho_4} \left(f^2\hat{R}_{\rho_1\rho_2}{}^{\sigma_1\sigma_2} + (\alpha - (\partial_z f)^2)\delta_{\rho_1\rho_2}^{\sigma_1\sigma_2} \right) \left(f^2\hat{R}_{\rho_3\rho_4}{}^{\sigma_3\sigma_4} + (\alpha - (\partial_z f)^2)\delta_{\rho_3\rho_4}^{\sigma_3\sigma_4} \right). \quad (3.10)$$

This will be proportional to the trace of the field equations (3.7) for the D -dimensional metric $\hat{g}_{\mu\nu}$ if the function $f(z)$ satisfies the relation

$$\alpha - (\partial_z f)^2 = \beta f^2 \quad (3.11)$$

Plugging in the components of the Riemann tensor into the equations $\mathcal{G}^\mu{}_\nu = 0$ gives the equation

$$0 = \delta_{\nu\sigma_1\dots\sigma_4}^{\mu\rho_1\dots\rho_4} \left(f^2\hat{R}_{\rho_1\rho_2}{}^{\sigma_1\sigma_2} + (\alpha - (\partial_z f)^2)\delta_{\rho_1\rho_2}^{\sigma_1\sigma_2} \right) \left(f^2\hat{R}_{\rho_3\rho_4}{}^{\sigma_3\sigma_4} + (\alpha - (\partial_z f)^2)\delta_{\rho_3\rho_4}^{\sigma_3\sigma_4} \right) \quad (3.12)$$

$$+ 8\delta_{\nu\sigma_1\dots\sigma_3}^{\mu\rho_1\dots\rho_3} \left((\alpha - (\partial_z f)^2 + f\partial_z^2 f)\delta_{\rho_1}^{\sigma_1} \right) \left(f^2\hat{R}_{\rho_2\rho_3}{}^{\sigma_2\sigma_3} + (\alpha - (\partial_z f)^2)\delta_{\rho_2\rho_3}^{\sigma_2\sigma_3} \right)$$

The first of terms will vanish by virtue of the equations of motion (3.7) for the D -dimensional metric $\hat{g}_{\mu\nu}$ if the relation (3.11) is satisfied. The second term will vanish, if the function $f(x)$ also satisfies the additional relation

$$\alpha - (\partial_z f)^2 + f\partial_z^2 f = 0. \quad (3.13)$$

If we write the D and $(D + 1)$ -dimensional curvature parameters as $\beta = 1/l'^2$ and $\alpha = 1/l^2$ respectively, then both relations are satisfied if the conformal factor $f(z)$ in the $(D + 1)$ -dimensional metric (3.2) is taken to be

$$f(z) = \frac{l'}{l} \sin(z/l'). \quad (3.14)$$

Note that we can take the $l' \rightarrow \infty$ limit of this result in which the curvature parameter $\beta = 1/l'^2$ of the D -dimensional metric vanishes. Our previous result for this case, that $f(z) = z/l$, is then seen as the small angle approximation to the more general result above. We also note that, for the sake of definiteness, we have focused on LUV theories with AdS vacua. However, the above result is only simply modified for LUV theories with deSitter vacua. If we take $\alpha = -1/l^2$ and $\beta = -1/l'^2$ then we find that the conformal scaling function $f(z)$ must be given by

$$f = \frac{l'}{l} \sinh(z/l'). \quad (3.15)$$

Finally, although we have again focused on LUV theories of order $p = 2$, these results hold also for LUV theories of higher order. The extension of the forms of equations (3.10) and (3.12) to higher order theories follows the discussion given above, with the result that the function $f(z)$ must again satisfy equations (3.11) and (3.13).

4. Concluding remarks

We have shown that the construction of black string and brane solutions in pure Lovelock gravity theories and in LUV theories parallels the simple constructions in pure Einstein theory and in Einstein theory with a nonzero cosmological constant respectively. While this does not fully solve the problem of constructing black brane solutions in Lovelock gravity theories, it represents useful progress in a number of ways.

On one hand, it has been argued in reference [12] that LUV theories represent the most physically interesting class of Lovelock gravity theories. From this perspective, our results cover precisely this most interesting case. Further, the simplicity of our results, and their parallels with Einstein gravity, suggest that there may well be other methods of constructing solutions in Einstein gravity that have simple generalizations to pure Lovelock and LUV theories. In particular, as we commented above, the appearance of string solutions with deficit solid angle on odd dimensional spheres, suggests that we might be able to find analogues of the Weyl solutions and the C-metric in certain of these theories.

On the other hand, one's primary interest may be in corrections to solutions to pure Einstein gravity due to higher curvature Lovelock terms. In this case, our results are limited, but provide some clues towards finding black string and brane solutions in the general case. Consider the black hole solutions of Gauss-Bonnet gravity which have the form (2.5) with the metric function $f(z)$ given in equation (2.6). Take $\alpha_2 = 0$, so that we have an asymptotically flat solution. For large r this approaches Schwarzschild as in equation (2.7). As $r \rightarrow 0$, however, where the curvature becomes large, the solution matches onto a solution to pure Gauss-Bonnet gravity. If we want to construct a black string solution, in this case, it should asymptote for large r to Schwarzschild with an extra flat direction, the black string of pure Einstein gravity. Our results suggest that it should also look simple at small radii. There it should approach the pure Gauss-Bonnet limit of (2.6) again with a flat direction added. For intermediate radii, the solution will presumably depend on the coordinate along the string in a non-trivial way.

Note added: we would like to draw the readers attention to reference [20], which appeared very shortly after the original version of this paper and presents an alternate route to many of the same results. In addition, the authors of [20] note that interesting exotic black brane solutions may be constructed in pure Gauss-Bonnet theory (and very similarly in higher order pure Lovelock theories) by adding flat directions to *any* 4-dimensional metric having vanishing Euler density.

Acknowledgments

The authors would like to thank the organizers of the *Scanning New Horizons: GR Beyond 4 Dimensions* workshop at the KITP where this work was carried out. DK would like to thank Fay Dowker, Satyanarayan Mohapatra and Jennie Traschen for helpful discussions on this subject. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949 and Grant No. PHY0244801 and by the Natural Sciences & Engineering Research Council of Canada.

References

- [1] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837 [[hep-th/9301052](#)].
- [2] B. Kol, *The phase transition between caged black holes and black strings: a review*, *Phys. Rept.* **422** (2006) 119 [[hep-th/0411240](#)].
- [3] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, *Ann. Phys. (NY)* **172** (1986) 304.
- [4] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498.
- [5] D.G. Boulware and S. Deser, *String generated gravity models*, *Phys. Rev. Lett.* **55** (1985) 2656.
- [6] J.T. Wheeler, *Symmetric solutions to the Gauss-Bonnet extended Einstein equations*, *Nucl. Phys. B* **268** (1986) 737.
- [7] J.T. Wheeler, *Symmetric solutions to the maximally Gauss-Bonnet extended Einstein equations*, *Nucl. Phys. B* **273** (1986) 732.
- [8] C. Barcelo, R. Maartens, C.F. Sopuerta and F. Viniegra, *Stacking a 4D geometry into an Einstein-Gauss-Bonnet bulk*, *Phys. Rev. D* **67** (2003) 064023 [[hep-th/0211013](#)].
- [9] T. Kobayashi and T. Tanaka, *Five-dimensional black strings in Einstein-Gauss-Bonnet gravity*, *Phys. Rev. D* **71** (2005) 084005 [[gr-qc/0412139](#)].
- [10] C. Sahabandu, P. Suranyi, C. Vaz and L.C.R. Wijewardhana, *Thermodynamics of static black objects in d dimensional Einstein-Gauss-Bonnet gravity with D_4 compact dimensions*, *Phys. Rev. D* **73** (2006) 044009 [[gr-qc/0509102](#)].
- [11] A. Chamblin, S.W. Hawking and H.S. Reall, *Brane-world black holes*, *Phys. Rev. D* **61** (2000) 065007 [[hep-th/9909205](#)].
- [12] J. Crisostomo, R. Troncoso and J. Zanelli, *Black hole scan*, *Phys. Rev. D* **62** (2000) 084013 [[hep-th/0003271](#)].
- [13] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Finite action principle for Chern-Simons AdS gravity*, *JHEP* **06** (2004) 036 [[hep-th/0405267](#)].
- [14] M. Bañados, R. Olea and S. Theisen, *Counterterms and dual holographic anomalies in CS gravity*, *JHEP* **10** (2005) 067 [[hep-th/0509179](#)].
- [15] R.C. Myers and J.Z. Simon, *Black hole thermodynamics in Lovelock gravity*, *Phys. Rev. D* **38** (1988) 2434.
- [16] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for even dimensional asymptotically AdS gravity theories*, *Phys. Rev. D* **62** (2000) 044002 [[hep-th/9912045](#)].
- [17] M.H. Dehghani, N. Bostani and A. Sheikhi, *Counterterm method in Lovelock theory and horizonless solutions in dimensionally continued gravity*, [hep-th/0603058](#).
- [18] R. Emparan and H.S. Reall, *Generalized Weyl solutions*, *Phys. Rev. D* **65** (2002) 084025 [[hep-th/0110258](#)].
- [19] C. Charmousis and R. Gregory, *Axisymmetric metrics in arbitrary dimensions*, *Class. and Quant. Grav.* **21** (2004) 527 [[gr-qc/0306069](#)].

- [20] G. Giribet, J. Oliva and R. Troncoso, *Simple compactifications and black p-branes in Gauss-Bonnet and Lovelock theories*, hep-th/0603177.